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# Complexity of Approximation by Conic Splines (Extended Abstract)

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## Abstract

In this paper we show that the complexity, i.e., the number of elements, of a parabolic or conic spline approximating a sufficiently smooth curve with non-vanishing curvature to within Hausdorff distance  $\varepsilon$  is  $c_1\varepsilon^{-1/4} + O(1)$ , or  $c_2\varepsilon^{-1/5} + O(1)$ , respectively. The constants  $c_1$  and  $c_2$  are expressed in the Euclidean and affine curvature of the curve. We also prove that the Hausdorff distance between a curve and an optimal conic arc tangent at its endpoints is increasing with its arc-length, provided the affine curvature along the arc is monotone. We use this property in a simple bisection algorithm for computing an optimal parabolic or conic spline.

## 1 Introduction

**Complexity of conic approximants.** We show that the complexity—the number of elements—of an optimal *conic spline* approximating a sufficiently smooth curve to within Hausdorff distance (See [6] for a definition)  $\varepsilon$ , is of the form  $c_1\varepsilon^{-\frac{1}{5}} + O(1)$ , where we express the value of the constant  $c_1$  in terms of the Euclidean and affine curvature (See Corollary 2). An optimal *parabolic spline* approximates a curve to fourth order, so its complexity is of the form  $c_2\varepsilon^{-\frac{1}{4}} + O(1)$ . Also in this case the constant  $c_2$  is expressed in the Euclidean and affine curvature. These bounds are obtained by first deriving an expression for the Hausdorff distance of a conic arc that is tangent to a (sufficiently short) curve at its endpoints, and that minimizes the Hausdorff distance among all such bitangent conics. Applying well-known methods like those of [2] it follows that this Hausdorff distance is of fifth order in the length of the curve, and of fourth order if the conic is a parabola. We derive explicit constants in these asymptotic expansions in terms of the Euclidean and affine curvature of the curve.

**Algorithmic issues.** For curves with monotone affine curvature, called *affine spirals*, we consider conic arcs

tangent to the curve at its endpoints, and show that among such *bitangent conic arcs* there is a unique one minimizing the Hausdorff distance. This optimal bitangent conic arc  $C_{\text{opt}}$  intersects the curve at its endpoints and at one interior point, but nowhere else. If  $\alpha : I \rightarrow \mathbb{R}^2$  is an affine spiral, its displacement function  $d : I \rightarrow \mathbb{R}$  measures the signed distance between the affine spiral and the optimal bitangent conic along the normal lines of the spiral. This displacement function has an *equioscillation property* (See Section 3) and the Hausdorff distance between a section of an affine spiral and its optimal approximating bitangent conic arc is a monotone function of the arc length of the spiral section. This useful property gives rise to a bisection based algorithm for the computation of an optimal interpolating tangent continuous conic spline. The scheme reproduces conics. We implemented such an algorithm, and compare its theoretical complexity with the actual number of elements in an optimal approximating parabolic or conic spline.

**Related work.** In [4] Fejes Tóth considers the problem of approximating a convex  $C^2$ -curve  $C$  in the plane by an inscribed  $n$ -gon. Fejes Tóth proves that, with regard to the Hausdorff distance, the optimal  $n$ -gon  $P_n$  satisfies  $\delta_H(C, P_n) = \frac{1}{8} \left( \int_0^l \kappa^{1/2}(s) ds \right)^2 \frac{1}{n^2} + O(\frac{1}{n^4})$ . Here  $l$  is the length of the curve,  $s$  its arc length parameter, and  $\kappa(s)$  its curvature. Ludwig [8] extends this result by deriving the second term in this asymptotic expansion.

These problems fall in the context of geometric Hermite interpolation, in which approximation problems for curves are treated independent of their specific parametrization. The seminal paper De Boor, Höllig and Sabin [2] also fits in this context. Floater [5] gives a method that, for any conic arc and any odd integer  $n$ , yields a geometric Hermite interpolant with  $2n$  contacts, counted with multiplicity. This scheme gives a  $G^{n-1}$ -spline, and has approximation order  $O(h^{2n})$ , where  $h$  is the length of the conic arc. Degen [3] presents an overview of geometric Hermite interpolation, also emphasizing differential geometry aspects.

**Overview.** Section 2 reviews some notions from affine differential geometry that we use in this paper. Section 3 introduces affine spirals, a class of curves

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which have a unique optimal bitangent conic. These optimal bitangent conic arcs have some nice properties giving rise to a bisection algorithm for their computation. The complexity analysis of optimal parabolic and conic splines is presented in Section 4. Section 5 presents the output of the algorithm in a specific example.

## 2 Mathematical preliminaries

Circular arcs and straight line segments are the only regular smooth curves in the plane with constant Euclidean curvature. Conic arcs are the only smooth curves in the plane with constant *affine curvature*. The latter property is crucial for our approach, so we briefly review some concepts and properties from affine differential geometry of planar curves. See also Blaschke [1].

**Affine curvature.** Recall that a regular curve  $\alpha : J \rightarrow \mathbb{R}^2$  defined on a closed real interval  $J$ , i.e., a curve with non-vanishing tangent vector  $T(u) := \alpha'(u)$ , is parametrized according to Euclidean arc length if its tangent vector  $T$  has unit length. In this case, the derivative of the tangent vector is in the direction of the unit normal vector  $N(u)$ , and the Euclidean curvature  $\kappa(u)$  measures the rate of change of  $T$ , i.e.,  $T'(u) = \kappa(u)N(u)$ . Euclidean curvature is a differential invariant of regular curves under the group of rigid motions of the plane, i.e., a regular curve is uniquely determined by its Euclidean curvature, up to a rigid motion.

The larger group of *equi-affine transformations* of the plane, i.e., linear transformations with determinant one (in other words, area preserving linear transformations), also gives rise to a differential invariant, called the *affine curvature* of the curve. To introduce this invariant, let  $I \subset \mathbb{R}$  be an interval, and let  $\gamma : I \rightarrow \mathbb{R}^2$  be a smooth, regular plane curve. The curve  $\gamma$  is parametrized according to *affine arc length* if

$$[\gamma'(r), \gamma''(r)] = 1. \quad (1)$$

Here  $[v, w]$  denotes the determinant of the pair of vectors  $\{v, w\}$ . It follows from (1) that  $\gamma$  has non-zero Euclidean curvature. Conversely, every curve  $\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}^2$  with non-zero Euclidean curvature satisfies  $[\alpha'(u), \alpha''(u)] \neq 0$ , for  $u \in J$ , so it can be reparametrized according to affine arc length.

Note that the property of being parametrized according to affine arc length is an invariant of the curve under equi-affine transformations. If  $\gamma$  is parametrized according to affine arc length, then differentiation of (1) yields  $[\gamma'(r), \gamma'''(r)] = 0$ , so there is a scalar function  $k$  such that

$$\gamma'''(r) + k(r)\gamma'(r) = 0. \quad (2)$$

The quantity  $k(r)$  is called the *affine curvature* of the curve  $\gamma$  at  $\gamma(r)$ . A regular curve is uniquely determined by its affine curvature, up to an equi-affine transformation of the plane.

The affine curvature can be expressed in terms of the derivatives of  $\gamma$  up to and including order four. We refer to the full version of the paper for details.

At a point of non-vanishing Euclidean curvature there is a unique conic, called the *osculating conic*, having fourth order contact with the curve at that point (or, in other words, having five coinciding points of intersection with the curve). The affine curvature of this conic is equal to the affine curvature of the curve at the point of contact. Moreover, the contact is of order five if the affine curvature has vanishing derivative at the point of contact. (The curve has to be  $C^5$ .) In that case the point of contact is a *sextactic point*. See [1] for further details.

**Conics have constant affine curvature.** Solving the differential equation (2) shows that a curve of constant affine curvature is a conic arc. More precisely, a curve with constant negative affine curvature is a hyperbolic, parabolic, or elliptic arc iff its affine curvature is negative, zero, or positive, respectively.

## 3 Approximation of affine spirals

**Displacement function.** A *bitangent conic* of a regular curve  $\alpha : I \rightarrow \mathbb{R}^2$  is a conic arc which is tangent to  $\alpha$  at its endpoints, such that each normal line of  $\alpha$  intersects the conic arc in a unique point. Therefore, a bitangent conic has a parametrization  $\beta : I \rightarrow \mathbb{R}^2$  of the form  $\beta(u) = \alpha(u) + d(u)N(u)$ , where  $d : I \rightarrow \mathbb{R}$  is the *displacement function* of the conic arc. The Hausdorff distance between  $\alpha$  and a bitangent conic  $C$  is equal to

$$\delta_H(\alpha, C) = \max_{u \in I} |d(u)|. \quad (3)$$

There is a one-parameter family of bitangent conics, so the goal is to determine an *optimal bitangent conic*, i.e., a conic in this family that minimizes the Hausdorff distance.

**Equioscillation property.** An *affine spiral* is a regular curve without sextactic points, in other words, a curve with monotone affine curvature. Affine spirals have a *unique optimal bitangent conic*, which is tangent to the curve at its endpoints, and intersects the curve in one additional interior point, but at no other interior point. Moreover, the displacement function of this optimal bitangent conic has an *equioscillation property*: there are exactly two parameter values at which the maximum in (3) is attained. More precisely, there are  $u_+, u_- \in I$  such that  $d(u_+) = -d(u_-) = \delta_H(\alpha, C_{\text{opt}})$  and  $|d(u)| < \delta_H(\alpha, C_{\text{opt}})$  if  $u \neq u_{\pm}$ . The

points  $\alpha(u_-)$  and  $\alpha(u_+)$  are separated by the interior point of intersection of  $\alpha$  and  $C_{\text{opt}}$ . The optimal bitangent conic is the unique bitangent conic having this equioscillation property, a property that gives rise to a simple algorithm for computing it. See the full paper for details.

**Monotonicity of optimal Hausdorff distance.** If one endpoint of the affine spiral moves along the curve  $\alpha$ , the Hausdorff distance between the affine spiral and its optimal bitangent conic arc is *monotone* in the arc length of the affine spiral. More precisely, let  $\alpha : [u_0, u_1] \rightarrow \mathbb{R}^2$  be an affine spiral arc. For  $u_0 \leq u \leq u_1$ , let  $\alpha_u$  be the sub-arc between  $\alpha(u_0)$  and  $\alpha(u)$ , and let  $\beta_u$  be the (unique) optimal bitangent conic arc of  $\alpha_u$ . Then the Hausdorff-distance between  $\alpha_u$  and  $\beta_u$  is a monotonically increasing function of  $u$ .

This property gives rise to a bisection method for the computation of an optimal conic spline approximating a spiral arc to within a given Hausdorff distance. Section 5 presents the output of this algorithm in a specific example.

#### 4 Complexity of conic splines.

In this section our goal is to determine the Hausdorff distance of a conic arc of best approximation to an arc of  $\alpha$  of Euclidean length  $\sigma > 0$ , that is tangent to  $\alpha$  at its endpoints. If the conic is a parabola, these conditions uniquely determine the parabolic arc. If we approximate by a general conic, there is one degree of freedom left, which we use to minimize the Hausdorff distance between the the arc of  $\alpha$  and the approximating conic arc  $\beta$ .

The main result of this section gives an asymptotic bound on this Hausdorff distance.

##### Theorem 1 (Optimal Hausdorff distance)

Let  $\beta$  be a conic arc tangent at its endpoints to an arc of a regular curve  $\alpha$  of length  $\sigma$ , with non-vanishing Euclidean curvature.

1. If  $\alpha$  is a  $C^8$ -curve, and  $\beta$  is a parabolic arc, then the Hausdorff distance between these arcs has asymptotic expansion

$$\delta_H(\alpha, \beta) = \frac{1}{128} |k_0| \kappa_0^{\frac{5}{3}} \sigma^4 + O(\sigma^5), \quad (4)$$

where  $\kappa_0$  and  $k_0$  are the Euclidean and affine curvature of  $\alpha$  at one of its endpoints, respectively.

2. If  $\alpha$  is a  $C^9$ -curve, and  $\beta$  is a conic arc, then the Hausdorff distance between these arcs is minimized if the affine curvature of  $\beta$  is equal to the average of the affine curvatures of  $\alpha$  at its endpoints, up to quadratic terms in the length of  $\alpha$ . In this case the Hausdorff distance has asymptotic expansion

$$\delta_H(\alpha, \beta) = \frac{1}{2000\sqrt{5}} |k'_0| \kappa_0^2 \sigma^5 + O(\sigma^6), \quad (5)$$

where  $\kappa_0$  is the Euclidean curvature of  $\alpha$  at one of its endpoints, and  $k'_0$  is the derivative of the affine curvature of  $\alpha$  at one of its endpoints.

The proof of this result is quite involved, but the main idea is rather simple. Let  $\alpha : [0, \varrho] \rightarrow \mathbb{R}^2$  be parametrized according to affine arc length. In particular,  $\varrho$  is the affine arc length of  $\alpha$ . One can show that

$$\varrho = \kappa_0^{\frac{1}{3}} \sigma + O(\sigma^2). \quad (6)$$

The parabolic arc, which is bitangent to  $\alpha$  at  $\alpha(0)$  and  $\alpha(\varrho)$ , is an offset curve depending on  $\varrho$ . Therefore it has a parametrization  $u \mapsto \beta(u, \varrho)$  of the form

$$\beta(u, \varrho) = \alpha(u) + d(u, \varrho) N(u), \quad (7)$$

where the displacement function  $d$  is of the form  $d(u, \varrho) = u^2(u - \varrho)^2 D(u, \varrho)$ . Then

$$\delta_H(\alpha, \beta) = \max_{0 \leq u \leq \varrho} |d(u, \varrho)| = \frac{1}{16} \varrho^4 |D(0, 0)| + O(\varrho^5). \quad (8)$$

In the full paper we show that the affine curvature of a curve of the form (7) is of the form

$$k_\beta = k_0 + 8 \kappa_0^{-\frac{1}{3}} D(0, 0) + O(\varrho). \quad (9)$$

Since  $\beta$  is a parabolic arc, its affine curvature is zero, i.e.,  $k_\beta = 0$ . Combining (6), (8), and (9) yields the asymptotic expression for the Hausdorff distance between the curve and its bitangent parabolic arc as stated in the first part of the theorem. The proof of the second part is more involved, but follows the same line of reasoning.

The preceding result gives an asymptotic expression for the minimal number of elements of an optimal parabolic or conic spline in terms of the maximal Hausdorff distance.

##### Corollary 2 (Complexity of conic splines)

Let  $\alpha : [0, L] \rightarrow \mathbb{R}^2$  be a regular curve of length  $L$ , with non-vanishing Euclidean curvature parametrized by Euclidean arc length, and let  $\kappa(s)$  and  $k(s)$  be its Euclidean and affine curvature at  $\alpha(s)$ , respectively.

1. If  $\alpha$  is a  $C^8$ -curve, then the minimal number of arcs in a tangent continuous **parabolic** spline approximating  $\alpha$  to within Hausdorff distance  $\varepsilon$  is

$$N(\varepsilon) = c_1 \left( \int_0^L |k(s)|^{\frac{1}{4}} \kappa(s)^{\frac{5}{12}} ds \right) \varepsilon^{-\frac{1}{4}} (1 + O(\varepsilon^{\frac{1}{4}})), \quad (10)$$

where  $c_1 = 128^{-\frac{1}{4}} \approx 0.297$ .

2. If  $\alpha$  is a  $C^9$ -curve, then the minimal number of arcs in a tangent continuous **conic** spline approximating  $\alpha$  to within Hausdorff distance  $\varepsilon$  is

$$N(\varepsilon) = c_2 \left( \int_0^L |k'(s)|^{\frac{1}{5}} \kappa(s)^{\frac{2}{5}} ds \right) \varepsilon^{-\frac{1}{5}} (1 + O(\varepsilon^{\frac{1}{5}})), \quad (11)$$

where  $c_2 = (2000\sqrt{5})^{-\frac{1}{5}} \approx 0.186$ .

We only sketch the proof, and refer to the papers by McClure and Vitale [9] and Ludwig [8] for details about this proof technique in similar situations. Consider a small arc of  $\alpha$ , centered at  $\alpha(s)$ . Let  $\sigma(s)$  be its Euclidean arc length. Then the Hausdorff distance between this curve and a bitangent parabolic arc is  $\frac{1}{128} |k_0| \kappa_0^{\frac{5}{3}} \sigma(s)^4 + O(\sigma(s)^5)$ , cf Theorem 1. Therefore,

$$\sigma(s) = \sqrt[4]{128} |k(s)|^{-\frac{1}{4}} \kappa(s)^{-\frac{5}{12}} \varepsilon^{\frac{1}{4}} (1 + O(\varepsilon^{\frac{1}{4}})).$$

The first part follows from the observation that  $N(\varepsilon) = \int_{s=0}^L \frac{1}{\sigma(s)} ds$ . The proof of the second part is similar.

## 5 Implementation

We implemented an algorithm for the computation of an optimal parabolic or conic spline, based on the monotonicity property. For computing the optimal parabolic spline, the curve is subdivided into affine spirals. Then for a given maximal Hausdorff distance  $\varepsilon$ , the algorithm iteratively computes optimal parabolic arcs starting at one endpoint. At each step of this iteration the next breakpoint is computed via a standard bisection procedure, starting from the most recently computed breakpoint. The bisection procedure yields a parabolic spline whose Hausdorff distance to the subtended arc is  $\varepsilon$ . An optimal conic spline is computed similarly. The bisection step is slightly more complicated, since the algorithm has to select the optimal conic arc from a one-parameter family. Here the equioscillation property gives the criterion for deciding whether the computed conic arc is optimal.

**A Spiral Curve.** We present the results of our algorithm applied to the *spiral curve*, parametrized by  $\alpha(t) = (t \cos(t), t \sin(t))$ , with  $\frac{1}{6}\pi \leq t \leq 2\pi$ .

Table 1 gives the number of arcs computed by the algorithm, and the theoretical bounds on the number of arcs for varying values of  $\varepsilon$ , both for the parabolic and for the conic spline.

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$\varepsilon$	Parabolic		Conic	
	Exp.	Th.	Exp.	Th.
$10^{-1}$	5	3	3	2
$10^{-2}$	9	5	4	3
$10^{-3}$	15	9	6	5
$10^{-4}$	26	16	9	7
$10^{-5}$	46	28	13	11
$10^{-6}$	82	50	21	17
$10^{-7}$	145	88	32	26
$10^{-8}$	254	157	50	41

Table 1: The complexity of the parabolic spline and the conic spline approximating the Spiral Curve, comparing the theoretical complexity with the complexity measured in experiments, for various values of the maximal Hausdorff distance  $\varepsilon$ .

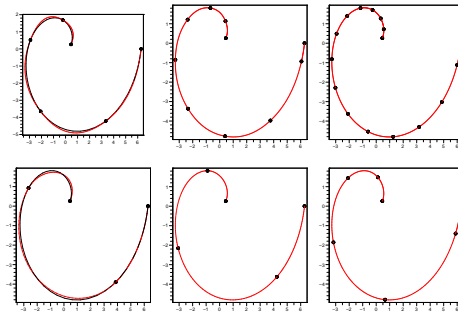


Figure 1: Row 1 shows the parabolic spline and row 2 shows conic spline approximation for the Spiral for  $\varepsilon = 10^{-1}, 10^{-2}$  and  $10^{-3}$

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